Catalytic Decoupling
Joint work with Mario Berta, Frédéric Dupuis, Renato Renner and Matthias Christandl
(arXiv:1605.00514, accepted for publication in PRL)

Deconstruction and Conditional Erasure of Correlations
Joint work with Mario Berta, Fernando Brandao, and Mark Wilde
(arXiv:1609.06994)

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Introduction:
Decoupling and Erasure
Decoupling

- Idea: destroy correlations by local noisy quantum channels
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- Proof tool in quantum Shannon theory, thermodynamics, solid state physics, black hole radiation...
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- bipartite quantum system $A \otimes E$ in mixed state $\rho_{AE}$
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$U_A(\cdot)U_A^\dagger$
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- $\log |A_2| \approx \frac{n}{2} I(A : E)_\sigma$ for $\rho = \sigma \otimes^n$ (Horodecki, Oppenheim, Winter ’05)
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⇒ Operational interpretation of the quantum mutual information!
Erasure of correlations

- Task introduced by Groisman, Popescu and Winter in '04

- bipartite quantum system $A \otimes E$ in mixed state $\rho_{AE}$
- Apply random unitary channel
- Correlations erased if approximately product
- How big do we have to choose $k$?
- optimal: $k \approx nI(A) \sigma$ for $\rho = \sigma \otimes n$
Erasure of correlations

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$$2^{-k} \sum_{i=1}^{2^k} U_i(\cdot)U_i^\dagger$$
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⇒ Operational interpretation of the quantum mutual information!

⇒ decoupling, erasure of correlations: two sides of same coin
Decoupling

\[ I(A : E)_\rho \]
Decoupling

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Catalytic Decoupling

\[ I^\varepsilon_{\text{max}}(A : E)_\rho \]
This talk

Decoupling

$$I(A : E)_\rho$$

Catalytic Decoupling

$$I^\varepsilon_{\text{max}}(A : E)_\rho$$

Conditional Erasure

$$I(A : E|R)_\rho$$

one-shot

side information

\(E\) \(\otimes\) \(A_1\)

\(\otimes\)

\(\otimes\)

\(\otimes\)

\(\otimes\)
Catalytic decoupling
Theorem (Dupuis, Berta, Wullschleger, Renner ’10)

Let $\rho_{AE}$ be a bipartite quantum state, and let $\mathcal{H}_A \cong \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ such that

$$\log |A_2| \geq \frac{1}{2} (H^\varepsilon_{\text{max}}(A)_\rho - H^\varepsilon_{\text{min}}(A|E)_\rho) - O \left(\log \frac{1}{\varepsilon}\right).$$

Then $\exists U_A$ such that

$$\left\| \text{tr}_{A_2} \left( U_A \rho_{AE} U_A^\dagger \right) - \frac{1}{|A_1|} \otimes \rho_E \right\|_1 \leq O(\varepsilon).$$
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Then $\exists U_A$ such that

$$\left\| \text{tr}_{A_2}\left(U_A\rho_{AE}U_A^\dagger\right) - \frac{1_{A_1}}{|A_1|} \otimes \rho_{E}\right\|_1 \leq \mathcal{O}(\varepsilon).$$
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but there are product states with

$H^{\epsilon}_{\text{max}}(A)_{\rho} - H^{\epsilon}_{\text{min}}(A|E)_{\rho} = O(\log |A|)$ \Rightarrow suboptimal for applications like state merging
One-shot decoupling

Theorem (Dupuis, Berta, Wullschleger, Renner ’10)

Let $\rho_{AE} = \sigma_{A'E'}^\otimes n$ be a bipartite quantum state, and let $\mathcal{H}_A \cong \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ such that

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One-shot state merging

- previous work on one-shot state merging:
One-shot state merging

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- One-shot coherent state merging possible with

\[ q(A : R)_\psi = \frac{1}{2} I_{\max}^\varepsilon (A : R) + \log \log |A| + O \left( \log \frac{1}{\varepsilon} \right) \] (Berta, Christandl, Renner '09)
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- tailored techniques
Generalize decoupling twofold:

- drop randomization condition
- allow free mixed ancillary states

\( A \otimes E \) in a mixed state \( \rho \)

- add ancillary system \( A' \) in a fixed state
- divide system \( A \otimes A' \) into two parts, \( \sim A_1 \otimes A_2 \)
- apply a unitary to \( AA' \)
- trace out \( A_2 \)

How big do we have to choose \( A_2 \) here?
Catalytic decoupling

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- how big do we have to choose $A_2$ here?
Theorem (CM, Berta, Dupuis, Renner, Christandl)

Let \( \rho \in B(H_A \otimes H_E) \) be a quantum state. Then, for any \( 0 \leq \varepsilon' < \varepsilon \), catalytic decoupling with error \( \varepsilon \) can be achieved with remainder system size \( \log |A_2| \approx \frac{1}{2} I_{\max}(E:A) \rho \).

Conversely catalytic decoupling is impossible whenever \( \log |A_2| < \frac{1}{2} I_{\max}(E:A) \rho \).

▶ max-mutual information:

\[
I_{\max}(A:B)_\rho = \min_{\sigma_B} D_{\max}(\rho_{AB} \parallel \rho_A \otimes \sigma_B)
\]

▶ \( D_{\max}(\rho \parallel \sigma) = \min_{\lambda \in \mathbb{R}} \{ \lambda \sigma \geq \rho \} \)

Two proofs, one using the techniques from Anshu et al. and Berta et al. respectively.
Theorem (CM, Berta, Dupuis, Renner, Christandl)

Let $\rho_{AE} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_E)$ be a quantum state. Then, for any $0 \leq \varepsilon' < \varepsilon$ catalytic decoupling with error $\varepsilon$ can be achieved with remainder system size

$$\log |A_2| \approx \frac{1}{2} I_{\max}^{\varepsilon'}(E : A)_{\rho}.$$ 

Conversely catalytic decoupling is impossible whenever

$$\log |A_2| < \frac{1}{2} I_{\max}^{\varepsilon}(E : A)_{\rho}.$$
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$$I_{\max}(A : B)_{\rho} = \min_{\sigma_B} D_{\max}(\rho_{AB} || \rho_A \otimes \sigma_B)$$
Characterization

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- Two proofs, one using the techniques from Anshu et al. and Berta et al. respectively
Denote the minimal remainder system size $\log |A_2|$ by $R^\varepsilon_c(A : E)_\rho$.

Minimal remainder system size if $\rho = \sigma \otimes n$:

$$R^\varepsilon_c(A : E)_\rho \approx \frac{1}{2} \left[ nI(A : E)\sigma + \sqrt{nV}I(A : E)\sigma \Phi^{-1}(\varepsilon) \right] + O(\log n)$$

Unitary randomizing and partial trace models equivalent with ancilla.
Properties

- Denote the minimal remainder system size \( \log |A_2| \) by 
  \( R_c^\varepsilon(A : E)_\rho \)

- Minimal remainder system size if \( \rho = \sigma \otimes \sigma^\otimes n \):
  \[
  \frac{1}{n} R_c^\varepsilon(A : E)_\rho \approx \frac{1}{2} I(A : E)_\sigma
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Properties

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  Asymptotically, the ancilla becomes unnecessary, usual randomization condition becomes redundant.
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  Asymptotically the ancilla becomes unnecessary, usual randomization condition becomes redundant
- Tightness of characterization allows derivation of a 2nd order term:
  \[
  R_c^\varepsilon(A : E)_\rho = \frac{1}{2} \left[ nI(A : E)_\sigma + \sqrt{nV_l(A : E)_\sigma} \Phi^{-1}(\varepsilon) \right] + \mathcal{O}(\log n)
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Properties

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- Unitary randomizing and partial trace models equivalent with ancilla.
Conditional Erasure
Erasure of conditional correlations

\[ \rho_{AER} \]
Erasure of conditional correlations

- $\rho_{AER}$
- Conditional quantum mutual information
  
  $$I(A : E | R)_\rho = H(\rho_{AR}) + H(\rho_{ER}) - H(\rho_{AER}) - H(\rho_{R})$$
Erasure of conditional correlations

- $\rho_{AER}$
- Conditional quantum mutual information
  \[ I(A : E|R)_\rho = H(\rho_{AR}) + H(\rho_{ER}) - H(\rho_{AER}) - H(\rho_R) \]
- Recoverability: if $I(A : E|R) = \varepsilon$ small,
  \[ \rho_{AER} \approx O(\varepsilon) \mathcal{R}_{R \rightarrow RA}(\rho_{ER}) \text{ for some quantum channel } \mathcal{R}. \]
  (Fawzi, Renner ’14)
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  (Fawzi, Renner ’14)
Erasure of conditional correlations

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- Conditional quantum mutual information
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$\Rightarrow$ All correlations of $A$ and $E$ mediated by $R$

$\Rightarrow$ $E - R - A$ is *approximate quantum Markov chain*

- $I(A : E|R)$ measures *conditional correlations*
- i.i.d. setting
- Recall: Erasure of correlations in $\rho_{AE}$ operating on $A$ costs $I(A : E)$ bits of noise.
Erasure of conditional correlations

$\rho_{AER}$

Conditional quantum mutual information

$I(A : E|R)_\rho = H(\rho_{AR}) + H(\rho_{ER}) - H(\rho_{AER}) - H(\rho_R)$

Recoverability: if $I(A : E|R) = \varepsilon$ small,

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$\Rightarrow$ All correlations of $A$ and $E$ mediated by $R$

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i.i.d. setting

Recall: Erasure of correlations in $\rho_{AE}$ operating on $A$ costs

$I(A : E)$ bits of noise.

Can we erase conditional correlations by injecting $I(A : E|R)_\rho$ bits of noise into $A$?
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Characterization for pure states: Noise $\gg I(A : E|R)$ necessary in general (Wakakuwa et al. ’15)
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\[ \exists \text{ Classical counterexample.} \]

- Characterization for pure states: Noise $\gg I(A : E|R)$ necessary in general (Wakakuwa et al. ’15)
- Obvious solution in the classical case: condition on $R$!
State redistribution (SRD)

Alice, Bob and a referee share a pure state $|\psi\rangle\langle\psi|$ AB

Alice has AC, Bob has B, Referee has R

their task: Alice has to send A to Bob

they can use entanglement

optimal communication rate $I(A: R | C)$ (Devetak and Yard '06)
Alice, Bob and a referee share a pure state $|\psi\rangle\langle\psi|_{ABCR}$.
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$\psi$
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Deconstruction, conditional erasure I

- State $\rho_{AER}$
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- State $\rho_{AER}$
- quantum conditional operation on $A$ conditioned on $R$: 

\[ \frac{A}{A'} \sim = \frac{A_1 A_2}{ \cdot} \]
- State $\rho_{AER}$
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allow ancilla like in catalytic decoupling
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Step-by-step definition:
Deconstruction, conditional erasure I

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Step-by-step definition:
- add ancillary system $A'$ in a fixed state
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Step-by-step definition:
- add ancillary system $A'$ in a fixed state
- apply a unitary $U_{RAA'}$

\[ U_{RAA'}(\cdot)U_{RAA'}^\dagger \]
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- trace out $A_2$
Different goals:

- Make $E - R - A_1$ an approximate quantum Markov chain,
- Deconstruction of correlations
- Make $A_1$ product with $ER$,
- Conditional erasure of correlations

Theorem (Berta, Brandao, CM, Wilde)

Conditional erasure of correlations is equivalent to quantum state redistribution. Asymptotically, deconstruction needs at least a rate of $I(A : E | R)$ bits of noise.

Both tasks have same optimal rate $I(A : E | R)$ of noise asymptotically.

Operational interpretation of quantum conditional mutual information!
Deconstruction, conditional erasure II

- Different goals:
  - make $E - R - A_1$ an approximate quantum Markov chain, *deconstruction* of correlations

\[ \begin{align*}
E & \approx \epsilon \\
\end{align*} \]

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Decoupling

$I(A : E)_\rho$

one-shot

Catalytic Decoupling

$I_{\text{max}}^\varepsilon (A : E)_\rho$

side information

Conditional Erasure

$I(A : E|R)_\rho$

simple one-shot state merging

operational interpretations of quantum discord and squashed entanglement
backup slides
One-shot coherent state merging (Berta et al. ’09)

Alice, Bob and a referee share a quantum state $|\psi\rangle\langle\psi|_{ABR}$. Their task: Alice has to send her part of the state to Bob. Alice needs ancilla – give purification to Bob ⇒ entangled resource!
Application

One-shot coherent state merging (Berta et al. ’09)

- Now: easy!
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![Diagram showing the quantum state $|\psi\rangle\langle\psi|_{ABR}$ among Alice, Bob, and the referee R.](image)
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$\Rightarrow$ one-shot state merging possible with $\frac{1}{2} I_{\max}^{\varepsilon}(A : R)$ qbits of communication
Applications

- 2-party state $\rho_{AB}$, measurement $\Lambda_{A\rightarrow X}$
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- (unoptimized) quantum discord:
  $$D(A : B)_{\rho,\Lambda} = I(A : B)_{\rho} - I(X : B)_{\Lambda(\rho)}$$

- Original interpretation: decrease of correlations under interaction with environment ("einselection", Zurek '00)

Theorem (Berta, Brandao, CM, Wilde)
$$D(A : B)_{\rho,\Lambda}$$ is equal to the rate of noise necessary to simulate the loss of correlations incurred by $\rho \otimes n$ under the action of $\Lambda \otimes n$.

- Squashed entanglement:
  $$E_{sq}(A : B)_{\rho} = \inf \sigma I(A : B | E)_{\sigma}, \inf \text{ over all } \sigma_{ABE} \text{ with } \text{tr} E_{\sigma_{ABE}} = \rho_{AB}$$

- Squashed entanglement is amount of noise necessary to make many i.i.d. copies of $\rho_{AB}$ close to separable by operation on $A$ and arbitrary catalytic side information $E$. 

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