The Thermality of Quantum Approximate Markov Chains
with implications to the Locality of Edge States and Entanglement Spectrum

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Motivation

When many-body systems are described by **local (short-range) Hamiltonians**, states have special correlation properties.

**Area law for gapped ground states**: restricts entanglement (rigorously proven for 1D systems [Hastings, 07])

**Area law for Gibbs (thermal) states**: restricts correlations (proven for any dim. [Wolf, et al., 07])

→ efficient descriptions of many-body states (MPS, PEPS, MPO,...)

A useful consequence of area laws: small “conditional mutual information (CMI)” on certain regions (Applications: [Kim, ‘12,’13], [Swingle & Kim, 14], [Kastryano & Brandao, ‘16] ...)

Q. How to characterize?
Motivation

When many-body systems are described by local (short-range) Hamiltonians, states have special correlation properties.

Area law for gapped ground states: restricts entanglement
(proven rigorously for 1D systems [Hastings, 07])

Area law for Gibbs (thermal) states: restricts correlations
(proven for any dim. [Wolf, et al., 07])

A useful consequence of area laws:
small “conditional mutual information (CMI)” on certain regions
(Application: [Kim, ‘12,’13], [Swingle & Kim, 14], [Kastryano & Brandao, ‘16] ...)

This talk:
≈ approximate Markov chains

1. Characterizing states with small CMI in terms of Gibbs states
(cf. previous talk by Kastoryano)

2. An application to “entanglement spectrum” of 2D gapped systems
   efficient descriptions of many-body states (MPS, PEPS, MPO, ...)

Q. How to characterize?
Outline of this talk

Part I: A characterization of approximate Markov chains
◆ Area law for Gibbs States
◆ Quantum Markov Chains & Approximate Quantum Markov Chains
◆ Equivalence to Gibbs states of short-range Hamiltonians

Part II: An application to entanglement spectrum in 2D systems
◆ Topological Entanglement Entropy and Entanglement Spectrum
◆ Previous Results on Entanglement Spectrum
◆ Locality of Entanglement Hamiltonian and Spectrum
Part I:
A characterization of approximate Markov chains
**Area law for Gibbs states**

Hamiltonian

\[
H = \sum_i h_{i,i+1}, \quad \| h_i \| \leq J.
\]

Gibbs state

\[
\rho = \frac{1}{Z} e^{-\beta H}, \quad Z = \text{tr} e^{-\beta H}.
\]

[Wolf, et al., '07]

\[
I(A: B)_\rho := S(A)_\rho + S(B)_\rho - S(AB)_\rho \leq 2\beta J |\partial A|
\]

\[
\Rightarrow S(A)_\rho := -\text{tr} \rho_A \log_2 \rho_A
\]
The conditional mutual information:

\[ I(A: C | B)_\rho := I(A: BC)_\rho - I(A: B)_\rho \geq 0 \]

- Monotonicity of MI: \( I(A: BC)_\rho \geq I(A: B)_\rho \)

\[ \rightarrow I(A: B_1)_\rho \leq I(A: B_1 B_2)_\rho \leq \cdots \leq I(A: B_1 \ldots B_m)_\rho \leq 2\beta J|\partial A| \]

small for large \( m \)!
Quantum Markov Chain (for three systems)

If $I(A: C|B)_{\rho} = 0$, quantum state $\rho_{ABC}$ is called a **Quantum Markov Chain** $A - B - C$.

1. There exists a CPTP-map $\Lambda_{B\to BC} : B \to BC$ s.t.
   \[
   \rho_{ABC} = \text{id}_A \otimes \Lambda_{B\to BC}(\rho_{AB})
   \]

2. There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.
   \[
   \rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0 \ (\rho_{ABC} > 0)
   \]

[Hayden, et al., 03], [Brown & Poulin, ‘12]
Longer Chains

$\rho_A$ on the chain $A_1A_2 \ldots A_n$ is a (quantum) Markov chain if

$$I(A_1 \ldots A_{i-1}: A_{i+1} \ldots A_n | A_i)_\rho = 0$$

for arbitrary $i \in [n]$.

*We can generalize the concept of Markov chains to general graphs as Markov networks*
Hammersley-Clifford Theorem (1D)

[Hammersley&Clifford, ‘71]:
Random variables $X_1, X_2, ..., X_n$ forms a (positive) Markov chain if, and only if, the distribution can be written as

$$p_{X_1X_2...X_n}(x_1, x_2, ..., x_n) = \frac{1}{Z} \exp \left( - \sum_i h_i(x_i, x_{i+1}) \right)$$

* also holds for Markov networks

Positive Markov chains

↓

Gibbs distributions of 1D short-range Hamiltonians
[Leifer & Poulin, ’08], [Brown & Poulin, ’12]:

A quantum state $\rho_{A_1\ldots A_n} > 0$ on a chain forms a Markov chain if, and only if, the state can be written as

$$\rho_{A_1\ldots A_n} = \frac{1}{Z} \exp \left( - \sum_i h_{A_i A_{i+1}} \right), \quad [h_{A_i A_{i+1}}, h_{A_j A_{j+1}}] = 0$$

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A_1 A_2 \ldots A_n
```

* also holds for Markov networks

- Positive quantum Markov chains
- **2.** There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.
  $$\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0$$

Gibbs states of 1D **commuting** short-range Hamiltonians
[Leifer & Poulin, ’08], [Brown & Poulin, ’12]:

A quantum state $\rho_{A_1\ldots A_n} > 0$ on a chain forms a Markov chain if, and only if, the state can be written as

$$\rho_{A_1\ldots A_n} = \frac{1}{Z} \exp\left( - \sum_i h_{A_i A_{i+1}} \right), \quad [h_{A_i A_{i+1}}, h_{A_j A_{j+1}}] = 0$$

--

2. There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.

$$\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0$$

* also holds for Markov networks
How about states having small but non-zero CMI?

Naïve guess: all properties of Markov chains *approximately* hold for approximate Markov chains

Classical:

\[
I(X:Z|Y)_p = \min_{q:\text{Markov}} S(p_{XYZ} \parallel q_{XYZ})
\]

\[
I(X:Z|Y)_p \leq \varepsilon \iff p_{XYZ} \approx_{\varepsilon} q_{XYZ}
\]

However...

Quantum:

\[
I(A:C|B)_\rho \neq \min_{\sigma:\text{Markov}} S(\rho_{ABC} \parallel \sigma_{ABC}) \quad [\text{Ibinson, et al., '06}]
\]

\exists \text{ property of Markov chains which is invalid for approximate Markov chains}
Local Recoverability of States with Small CMI

Some properties still approximately hold for approximate Markov chains

[Fawzi & Renner, ‘15]:
There exists a CPTP-map $\Lambda_{B \rightarrow BC}$ s.t.

$$I(A: C | B)_\rho \geq -2 \log_2 F(\rho_{ABC}, \Lambda_{B \rightarrow BC}(\rho_{AB}))$$

$$I(A: C | B)_\rho \approx 0 \iff$$

1. There exists a CPTP-map $\Lambda_{B \rightarrow BC}: B \rightarrow BC$ s.t.

$$\rho_{ABC} \approx \text{id}_A \otimes \Lambda_{B \rightarrow BC}(\rho_{AB})$$

*The converse part can be shown by using the Alicki-Fannes inequality.*
Q. How about the quantum Hammersley-Clifford theorem for approximate Markov chains?
Approximate Quantum HC Theorem (1D)

Result 1.
For any $\varepsilon$-approximate Markov chain $\rho_{A_1A_2 \ldots A_n}$, there exists a Hamiltonian $H_A = \sum h_{A_iA_{i+1}}$ s.t.,

$$S(\rho_A || e^{-H_A}) \leq n\varepsilon.$$
Result 2.
For any Gibbs state $\rho$ of a short-range Hamiltonian $H$ at temperature $T$

$$I(A: C | B)_\rho \leq ce^{-q(T)\sqrt{l}}$$

for $q(T) = e^{-c'T^{1}}$, $c \geq 0$, $c' > 0$ and any partition $ABC$ as in the diagram.

Application to Gibbs state preparation (see previous talk)

All 1D Gibbs states of short-range Hamiltonians are approximate Markov chains
(Strengthen the area law of 1D Gibbs states)
Approximate Quantum HC Theorem (1D)

For any Gibbs state $\rho$ of a short-range Hamiltonian $H$ at temperature $T$,

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Application to Gibbs state preparation (see previous talk)

All 1D Gibbs states of short-range Hamiltonians are approximate Markov chains
(Strengthen the area law of 1D Gibbs states)
PartII:
An application to entanglement spectrum in 2D systems
Area Law in 2D Gapped Systems

- Ground states of 2D gapped local Hamiltonians typically obey area law:
  \[ S(A)_\rho = \alpha |\partial A| - n_{\partial A} \gamma + o(1) \]
  
  \( \gamma \): topological entanglement entropy
  \[ \text{[Kitaev & Preskill, '06], [Levin & Wen '06]} \]
  \( \gamma > 0 \leftrightarrow \text{the g.s. is in a topologically ordered phase (??)} \)

A strong type of area law (rest of this talk)

\[ S(A)_\rho = \alpha |\partial A| - n_{\partial A} \gamma + e^{-|\partial A|/\xi} \]

\[ I(A: C | B)_\rho \leq e^{-cl} \]

\( \rho_{ABC} \) is an approximate Markov chain
Entanglement Hamiltonian and Spectrum

- Other tools to study gapped g.s.

\[ \rho_A = e^{-H_A} \quad \text{(entanglement Hamiltonian)} \]

\[ \lambda(H_A): \text{entanglement spectrum} \]

(logarithm of the Schmidt coefficients)

- Correspondence to edge theory in FQHE [Li & Haldane, ’08]
  also has been studied in other systems [Ali, et al., ’09, Lauchli & Bergholtz, ’10,…]

- Previous observations in the PEPS formalism
  [Cirac et al., ’11], [Schuch, et al., ’13], [Cirac, et al., ’16]

\[ \rho_l = V \sigma_b^2 V^\dagger \quad V: \text{isometry} \]

\[ H_b = \begin{cases} 
\text{short-range} \\
\text{in trivial phase} \\
\text{short-range + global interactions} \\
\text{in topologically ordered phases} 
\end{cases} \]
Entanglement Hamiltonian and Spectrum

• Other tools to study gapped g.s.

\[ \rho_A = e^{-H_A} \quad \text{entanglement Hamiltonian} \]

\[ \lambda(H_A) : \text{entanglement spectrum} \]

(      )

(logarithm of the Schmidt coefficients)

– **This talk**: connection to the *topological entanglement entropy*

also has been studied in other systems [Ali, et al., ‘09, Lauchli & Bergholtz, ‘10, …]

– Previous observations in the PEPS formalism

[  ]

[  ]

[  ]

\[ \rho_l = V \sigma_b^2 V^\dagger \quad V: \text{isometry} \]

\[ H_b = \begin{cases} 
\text{short-range} \\
\text{(in trivial phase)} \\
\text{short-range + global interactions} \\
\text{(in topologically ordered phases)} 
\end{cases} \]

\[ \sigma_b^2 = e^{-H_b} \]

\[ 1D \text{ virtual edge} \]

Q. How general this observation in PEPS?

This talk: connection to the topological entanglement entropy
also has been studied in other systems [Ali, et al., ‘09, Lauchli & Bergholtz, ‘10, …]

– Previous observations in the PEPS formalism

[ Cirac et al., ‘11], [Schuch, et al., ‘13], [Cirac, et al., ‘16]
Locality of Entanglement Spectrum ($\gamma = 0$)

Suppose $|\psi_{YY'}\rangle$ satisfies the area law and $\gamma = 0$ (trivial phase).

$\rightarrow \rho_{X_1...X_m}$ is an approx. Markov chain

Result 1.

$\rightarrow \rho_{X_1...X_m} \approx \frac{1}{Z} \exp(-\sum h_{X_iX_{i+1}})$

- $|\psi_{YY'}\rangle$ is pure $\rightarrow \lambda(\rho_{YY'}) = \lambda(\rho_{X_1...X_m})$

- $I(Y:Y')_\rho = I(Y:Y'|X)_\psi \approx 0 \rightarrow \rho_{YY'} \approx \rho_Y \otimes \rho_{Y'} = \rho_Y \otimes 2$

assume reflection sym.

$H_Y^{(2)} := \log \rho_Y \otimes I + I \otimes \log \rho_Y$

$\Rightarrow \| \lambda \left( H_Y^{(2)} \right) - \lambda(\sum h_{X_iX_{i+1}}) \|_1 \leq e^{-cl}$ for some $c > 0$. 
How about the case of $\gamma > 0$?

**Result 3.**

Under our assumption, for some $c > 0$ and sufficiently large $l$,

$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X || e^{-H_X}) + e^{-cl} \geq 0 \ (l \gg 1)$$

$$\mathcal{H}_2 := \{ H = \sum h_{X_iX_{i+1}}, \| h_{X_iX_{i+1}} \| \leq O(|X|) \}$$

$\gamma > 0 \rightarrow -\log \rho_X$ is non-local

**Note:** EH is local after tracing out $X_i$.

$$\text{tr}_{X_1} e^{-H_X} = \exp(-h_{X_2X_3} \cdots - h_{X_{m-1}X_m})$$

**Conjecture (no rigorous proof):**
The non-local part is dominated by \textit{m-body} interactions
Non-Locality of Entanglement Spectrum ($\gamma > 0$)

Result 3.
Under our assumption, for some $c > 0$ and sufficiently large $l$,

$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X || e^{-H_X}) + e^{-cl}$$

$\mathcal{H}_2 := \{ H = \sum h_{X_iX_{i+1}}, \| h_{X_iX_{i+1}} \| \leq O(\|X\|) \}$

$$\| \lambda(H_Y^{(2)}) - \lambda(H_X) \|_1 \leq e^{-cl}$$

for a non-local $H_X$. 

Difference to The Previous Results

Assumption: PEPS formalism (fixed-point) [Cirac et al., ‘11], [Schuch, et al., ‘13], [Cirac, et al., ‘16]

\[ \lambda(-\log \rho_l) = \lambda(H_b) \]  

\[ H_b = \begin{cases} 
\text{short-range} & \text{(in trivial phase)} \\
\text{short-range + global interactions} & \text{(in topologically ordered phases)} 
\end{cases} \]

Assumption: Strong type of area law (+ reflection symmetry) this talk

\[ \| \lambda(H^{(2)}_Y) - \lambda(H_X) \|_1 \leq e^{-cl} \]  

\[ H_X = \begin{cases} 
\text{short-range} & (\gamma = 0) \\
\text{short-range + global interactions} & (\gamma > 0) 
\end{cases} \]
Take-home massages:
Part I: Quantum approximate Markov chains are Gibbs states of 1D short-range Hamiltonians.

Part II: The locality of the entanglement spectrum of gapped g.s. on a cylinder is related to the TEE.

Open problems:
Part I: Better bounds on CMI of 1D Gibbs states?
   Generalization of the equivalence to Markov networks?
   (→ application for Gibbs state preparation)

Part II: Weaker assumptions?
   Do we really need double of the ES?
   Consequences of the (non-)locality of ES?
THANK YOU!
Idea of the proof

Result 1.
For any $\varepsilon$-approximate Markov chain $\rho_{A_1A_2...A_n}$, there exists a Hamiltonian $H_A = \sum h_{A_iA_{i+1}}$ s.t.,

$$S(\rho_A \| e^{-H_A}) \leq n\varepsilon.$$

• **The maximum entropy principle** [Jaynes, ‘57]

  The maximum entropy state $\sigma_A$ satisfying
  $$\sigma_{A_iA_{i+1}} = \rho_{A_iA_{i+1}}, \forall i$$

  has the form
  $$\sigma_{A_iA_{i+1}} = e^{-\sum h_{A_iA_{i+1}}}.$$

• **A result from information geometry** [Knauf & Weis, ‘10]

  $$\inf_{H_A = \sum h_{A_iA_{i+1}}} S(\rho_A \| e^{-H_A}) = S(A)\rho - S(A)\sigma$$

  Small by the assumption + SSA
Idea of the proof

Result 2.
For any Gibbs state $\rho$ of a short-range Hamiltonian $H$ at temperature $T$,

$$I(A: C \,|\, B)_{\rho} \leq ce^{-q(T)\sqrt{l}}$$

for $q(T) = e^{-c'T^{-1}}$, $c \geq 0$, $c' > 0$ and any partition $ABC$ as in the below.

Explicitly construct a recovery map $\Lambda_{B \to BC}$ s.t.

$$\|\rho_{ABC} - \Lambda_{B \to BC}(\rho_{AB})\|_1 \leq c'e^{-q'\sqrt{l}}$$

- Quantum belief propagation equation [Hastings, ‘07][Kim, ‘11]

For 1D Hamiltonian with short-range $H$, $\exists O_I$ s.t.

$$\|e^{-\beta(H+V)} - O_I e^{-\beta H} O_I^\dagger\| \leq e^{-q''l}$$
Idea of the proof

From the quantum belief propagation equation, there exists $X_B$ s.t.

$$\rho_{ABC} \approx \kappa_{B \rightarrow BC}(\rho_{AB}) = X_B (\text{tr}_{BR} [X_B^{-1} \rho_{AB} (X_B^{-1})^\dagger] \otimes \rho_{BR_C}) X_B^\dagger$$
Repeat-until-success method

We normalize $\kappa_{B \rightarrow BC}$ and define a CPTD-map $\tilde{\Lambda}_{B \rightarrow BC}$.

$\rightarrow$ Succeed to recover with a constant probability $p$ (in 1D systems).

Choose $N \sim l \ (|B| = O(l^2))$.

We can construct a CPTP-map $\Lambda_{B \rightarrow BC}$ satisfying

$$\|\rho_{ABC} - \text{id}_A \otimes \Lambda_{B \rightarrow BC} (\rho_{AB})\|_1 \leq e^{-O(l)}.$$
Result 3.
Under our assumption, for some $c > 0$ and sufficiently large $l$,

$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X \| e^{-H} \rho_X) + e^{-cl}$$

$$\mathcal{H}_2 := \{ H = \sum h_{X_iX_{i+1}}, \| h_{X_iX_{i+1}} \| \leq O(|X|) \}$$

By assumption, $I(X_1: X_3X_{m-1}|X_2X_m)_\rho \approx 0$.

$\rightarrow \exists$ recovery map $\Lambda_{2m\rightarrow 12m}: X_2X_m \rightarrow X_2X_mX_1$

$$\sigma_X := \Lambda_{2m\rightarrow 12m}(\rho_{X_2...X_m})$$

Facts: $\sigma_{X_iX_{i+1}} \approx \rho_{X_iX_{i+1}}$

$\rightarrow \sigma_X \approx \arg\min_{H_X \in \mathcal{H}_2} S(\rho_X \| e^{-H} \rho_X), \quad S(\rho_X \| \sigma_X) \approx 2\gamma$. 