Symmetry protected topological order at nonzero temperature

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Motivation: phases of matter

- Want robust computational structures

Topological phases are fascinating and useful for quantum computation. The 2D Ising ferromagnet, a self-correcting classical memory, and Kitaev's toric code, a quantum error correcting code at $T=0$, are examples of such models. Finding models with topological order at $T > 0$ is an important problem.
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- Order in many body spin systems $\Rightarrow$ robust computational structures
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The question

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2. First result: thermal instability of a class of SPT models
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1. Introduction: what are (symmetry protected) topological phases?
2. First result: thermal instability of a class of SPT models
3. Second result: existence of thermal SPT order
What are topological phases of matter?

- Gapped Hamiltonian $H = \sum_i h_i$ with (geometrically) local terms

- Defining property: Ground space properties are robust to any small local perturbations
  1. Ground space is a quantum code! e.g. toric code, color code
  2. Information is encoded in nonlocal degrees of freedom
  3. Robust to local errors
  4. Often ground space degeneracy depends on boundary conditions (e.g. genus of surface)
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Ground states of topologically ordered systems

Def: $|\psi_1\rangle \sim |\psi_2\rangle$ (belong to same phase) iff they are related by a constant depth unitary circuit
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- Trivial phase = equivalence class of a product state
- Topologically ordered $\implies$ not equivalent to a product state.
Symmetry protected topological order: states

- Physical systems often have symmetries (e.g. invariance under spin flip $S = \otimes_{\nu} X_{\nu}$) that give rise to richer physics
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Example: SPT order in 1D

- Easiest example: 1D cluster state global \textit{onsite} symmetry.

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**Def** \( |\psi\rangle \) is SPT ordered if no *symmetric* constant depth circuit can map it to a product state, *unless* the symmetry is broken.
Generalized SPT models in $d$-dimensions

- A broad class of SPT models in $d$ dimensions are the so-called group cohomology models of Chen-Gu-Liu-Wen 13

\[ H = \sum_v h_v, \quad [h_v, h_w] = 0 \]

- Has a global symmetry that acts \textit{onsite}

\[ S(g) = \prod_{\text{sites}} u(g), \quad [S(g), H] = 0, \quad g \in G \]
Applications of SPT order

Codes:
- Gapped boundaries

SPT states

Codes:
- Fault tolerant gate

Measurement-based quantum computation

Fault tolerant gate

Gapped boundaries
Applications of SPT order

SPT states

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Measurement-based quantum computation

- 2D toric code
- 1D cluster
- No symmetry
- Symmetry
- $T = 0$
- $T > 0$

Question: What about all of these at nonzero temperature?
Applications of SPT order

SPT states

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<table>
<thead>
<tr>
<th>No sym</th>
<th>Symmetry</th>
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<tbody>
<tr>
<td>$T = 0$</td>
<td>$2D$ toric code</td>
</tr>
<tr>
<td>$T &gt; 0$</td>
<td>$4D$ toric code</td>
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The problem

- Do any of the ground state properties of an SPT ordered system survive at nonzero temperature?

Our results
1. We rule out thermal stability of a large class of SPT models.
2. Prove thermal SPT ordering of the 3D cluster model

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\[ \Rightarrow \text{Thermal resources for MBQC, stable domain walls at } T \geq 0, \ldots \]
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![Diagram](image)

→ Thermal resources for MBQC, stable domain walls at $T \geq 0$, ...

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**Def** We say $\rho$ is $(r, \epsilon)$ SPT-trivial if
  \[
  \|\rho - \text{Tr}_{\mathcal{H'}} (U \rho_{cl} U^\dagger)\|_1 < \epsilon,
  \]
  - $\rho_{cl}$ is the Gibbs state of a classical Hamiltonian on an enlarged space
  - $U$ is a symmetric circuit of depth $r$
  - $\mathcal{H}'$ is the ancillary space
First result: instability of global onsite models

**Result 1:** Theorem: For any $T>0$, SPT models protected by global \textit{onsite} symmetries are not thermally robust, i.e., they are $(r, \epsilon)$ SPT-trivial for

- $r = \mathcal{O} \left( \log \frac{d+1}{d} (L) \right)$
- $\epsilon = \text{poly}^{-1}(L)$

where $L$ is linear size of a $d$ dimensional lattice.
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First result: Thermal triviality

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Hamiltonian
\[ H = \sum_v h_v \]

Remove terms with probability \( p_\beta \)

\[ p_\beta = \frac{2}{1+e^{2\beta}} \]

- Ground space of \( H(p_\beta) \) approximates the Gibbs state of \( H \) up to \( \text{poly}^{-1}(L) \) error
First result: Thermal triviality

- Second technical tool: local disentangler

Can construct a symmetric disentangler near each missing term, e.g. for qubits $D_v^h v \sim X_v$...

...and continue: $D_v^h v_1 \sim X_{v_1}$

High probability of a missing term in each log $p_{L^{q\hat{q}}} \log_{\frac{1}{2}} p_{L^{q}}$ region $O_p \log_p L^{q\hat{q}}$ spins to disentangle with gates of range $\tilde{O}_p \log_{\frac{1}{2}} p_{L^{q\hat{q}}}$

This gives a low-depth preparation of the Gibbs ensemble.
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- Instability of the associated computational structures at $T > 0$?
- Beyond group cohomology?
Second result

- The existence of thermally stable SPT order
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**Result 2:** The Raussendorf-Bravyi-Harrington (RBH) cluster model in 3D belongs to a thermally stable SPT phase for $0 \leq T < T_c$
The Raussendorf-Bravyi-Harrington (RBH) model

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Parts of the computation work thermally up to a critical temperature but there is no thermodynamic phase transition! Let's explore in the context of SPT phases!
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  \[\rightarrow\text{ Lets explore in the context of SPT phases!}\]
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- Cubic lattice with qubits on edges and faces - RBH 05

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H_C = - \sum_u K_u
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K = \begin{bmatrix}
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  X \\
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- Unique ground state: \( K_u |\psi_C\rangle = |\psi_C\rangle \)

- Constant depth preparation: \( |\psi_C\rangle = \prod_{u,w} CZ_{u,w} |+\rangle^N \)
Generalized symmetries

- Generalized symmetry: $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry.

$$S_M(g) = \prod_{u \in M} X_u, \quad M \text{ a 2-dim surface}$$

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$$[H, S_M(g)] = 0$$
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- A symmetry for each sublattice
- Operators naturally arise in error correction for the topological MBQC scheme

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$$[H, S_{\mathcal{M}}(g)] = 0$$
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- Two ways of proving this:
  1. Explicit order parameters
  2. Gauging the model
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- Two ways of proving this:
  1. Explicit order parameters
     $\implies$ Measurement based quantum computation and error correction
  2. Gauging the model
     $\implies$ Domain wall in quantum error correcting code
Sheet order parameter

- Sheet order parameters: symmetry operators with ‘twisted boundaries’

If $\rho_{\text{triv}}$ is $p_r \epsilon_q$-trivial with $r \alpha L \{2$, then the expectation value of these membrane operators is small.
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If $\rho_{\text{triv}}$ is $(r, \epsilon)$-trivial with $r < L/2$, then the expectation value of these membrane operators is small

Compare with

$$\langle XX \rangle + \langle ZZ \rangle \leq 1$$

for product states
Error correction can maximize the expectation value of the RBH thermal state with these membranes.

- Excitations are string-like objects.
- Syndrome = boundaries of strings.
- Apply correction map to return to \(1\)-eigenspace of 1-form operators.
- Correction succeeds if no homologically nontrivial excitations.
- Closed loops that are boundaries commute with membrane operators!
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- Correction succeeds if no homologically nontrivial excitations
- Closed loops that are boundaries commute with membrane operators!
- This protocol succeeds below $T_c$ due to string tension of excitations
Operational features

- Operationally: sheet order parameter quantifies the ability to distil maximally entangled pairs (encoded in toric codes) using single qubit measurements.

\[
\langle XX \rangle = \langle ZZ \rangle = 1
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\[ \langle XX \rangle = \langle ZZ \rangle = 1 \]

- Definition of SPT is protocol independent as one can use optimal decoder i.e. maximum likelihood decoding.
Briefly: Generalized gauging

- Can define a 4D system with boundary, that is 1-form symmetric

\[ H = H_{\text{bulk}}^{4D} + H_{\text{boundary}}^{3D} \]

- Gauging gives 4D toric code with domain wall:
  - Exchanges 1D loop-like electric and magnetic excitations

\[ e_1 \leftrightarrow m_2 \quad m_1 \leftrightarrow e_2 \]
Conclusion: in this talk

1. Thermal fragility of SPT models protected by global onsite symmetries
2. Robustness of SPT in the 3D cluster scheme
3. Computational aspects (distilling entanglement, fault tolerant gates, error correction)

   Usefulness of SPT for measurement based quantum computation with 1-form symmetry

- Steps toward understanding what is possible: thermally stable computational phases of matter
Further questions

1. The relationship between thermal SPT non triviality and computational power (in MBQC)
   \[\Rightarrow\] Analogous to the question of thermal topological order and its relationship to self-correcting quantum memories

2. Interesting topological defects in 3D

3. Symmetry principles for the single-shot error correction in 3D gauge color code

4. More models: interplay with transversality, symmetry enriched topological phases